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AN EFFECTIVE ARMA MODELING METHOD

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ABSTRACT

The ability to model random time series plays a prominent role in a variety of applications as exemplified by seismic data analysis, doppler radar processing, speech processing, adaptive filtering, and, array processing. Undoubtedly, two of the more popular procedures for effecting such time series models are the classical Fourier (MA) approach and the maximum entropy (AR) method. In this paper, a theoretical comparison of these contemporary procedures with a more general ARMA method will be made. It will be demonstrated that the spectral estimation performance of the ARMA method typically exceeds that of its more specialized MA and AR counterparts. With this supremacy thus established, a recently developed method for effectively generating ARMA model estimates from time series observations will be then presented.

I. INTRODUCTION

A problem which arises in a variety of applications is that of estimating the statistical characteristics of a random wide-sense stationary time series $\{x(n)\}$. This estimation is typically based upon a finite set of time series observations. In many signal processing applications, only the second order statistics as represented by the time series' autocorrelation sequence

$$r_x(n) = E\{x(n+m)x^*(m)\} \tag{1}$$

is required. In this expression, the symbols E and * denote the operations of expectation and complex conjugation, respectively. Upon taking the Fourier transform of this deterministic autocorrelation sequence, we obtain the associated power spectral density function

$$S_x(e^{j\omega}) = \sum_{n=-\infty}^{\infty} r_x(n)e^{-j\omega n} \tag{2}$$

It often happens that the essential attributes of a time series are more discernible from its frequency domain spectral density function than from its equivalent time domain autocorrelation sequence. It is with this in mind that interest in spectral estimation techniques has evolved.

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In the classical estimation problem, it is desired to achieve an estimation of the spectral density function (2) from observations of a time series. Without loss of generality, these observations will be here taken to be the N contiguous elements

$$x(1), x(2), \ldots, x(N) \tag{3}$$

A variety of procedures have been proposed for using these observations to achieve a spectral density estimate. Without doubt, the overwhelming number of procedures ultimately result in a rational spectral density model which fits the form

$$\hat{S}_x(e^{j\omega}) = \left| \frac{b_0 + b_1 e^{-j\omega} + \ldots + b_q e^{-jq\omega}}{1 + a_1 e^{-j\omega} + \ldots + a_p e^{-jp\omega}} \right|^2 = \left| \frac{B_q(e^{j\omega})}{A_p(e^{j\omega})} \right|^2 \tag{4}$$

The a_k and b_k coefficients of this model are referred to as its autoregressive and moving average coefficients, respectively. This model is commonly referred to as an autoregressive-moving average (ARMA) spectral model of order (p,q) where q and p denote the orders of the numerator and denominator polynomials, respectively. It is readily shown that any continuous (in ω) spectral density may be approximated arbitrarily closely by the above rational model if the order (p,q) is selected suitably large. Thus, the robustness of this rational model is apparent.

In studies directed towards spectral analysis, the preponderance of effort has been directed towards two special cases of the general ARMA model (4). They are the moving average (MA) model for which $A_p(e^{j\omega}) \equiv 1$, and, the autoregressive (AR) model for which $B_q(e^{j\omega}) = b_0$. The spectral density arising from a MA model is seen to contain no poles, and, as such it is known as an all-zero model. Similarly, the AR model is referred to as an all-pole model, and, the general ARMA model is seen to be a pole-zero model. Undoubtedly, the primary reasons for interest in the special case MA and AR models are that they:

(i) are amenable to a tractable analysis.
(ii) typically provide adequate spectral estimation performance.

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In this report, the effectiveness of AR, MA, and ARMA spectral models is examined. It is shown that, in general, the recently developed high performance ARMA modeling method provides superior spectral estimation performance.

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(iii) are synthesizable by efficient algorithms.

Despite these factors, it is widely recognized that the more robust ARMA model typically provides a much superior spectral estimation performance and uses fewer model parameters in the estimate. The main impediment to its use on a wider scale has been the lack of a specific procedure for obtaining the ARMA model parameters in a computationally efficient manner. Recently, an ARMA spectral modeling method which possesses this computational capability has been developed [1]-[4]. The main features of this new procedure are outlined in Sections IV and VI.

In the next three sections, theoretical procedures for generating MA, AR, and ARMA models from a finite set of actual autocorrelation values are presented. This is then followed by an example section which treats the classical problem of spectral estimating a time series composed of two sinusoids in additive white noise. It is there shown that the more general ARMA modeling procedure easily outperforms its special case MA and AR modeling procedures. With the ARMA spectral modeling methods superior performance thereby demonstrated for this idealistic situation (i.e., actual autocorrelation values are given), we next direct our attention to the more practical problem of generating ARMA spectral estimates from a finite set of observations (3). In particular, the recently developed high performance ARMA modeling method is outlined [2]&[4]. This novel procedure has been found to outperform such alternate ARMA modeling procedures as the Box-Jenkins method [10] and whitening filter approaches [11]&[12].

II. MA SPECTRAL MODELING

There exist a variety of procedures for obtaining a MA model of a wide sense stationary time series. These include the periodogram and different versions of the autocorrelation method. In each case, the spectral estimate will be of the form

$$\hat{S}_x(e^{j\omega}) = \left| b_0 + b_1 e^{-j\omega} + \ldots + b_q e^{-jq\omega} \right|^2 \qquad (5)$$

This MA spectral model requires a rather large value for the order parameter q to enable it to achieve a desirable frequency resolution performance. Unfortunately, this requirement can lead to a rather poor spectral estimation behavior in the case of moderate length time series observations. This behavior will be illustrated in the numerical example section.

In this section, we will be concerned with evolving a MA spectral estimate procedure for the special case in which one has available the q+1 autocorrelation elements $r_x(0), r_x(1), \ldots, r_x(q)$. With this information provided, a standard estimation is generated by the truncated series [5].

$$\hat{S}_x(e^{j\omega}) = \sum_{n=-q}^{q} w(n) r_x(n) e^{-j\omega n} \qquad (6)$$

in which w(n) is a symmetric window function which is selected to effect some desired behavior. In the pure truncated case, the rectangular window is used in which case $w(n) = 1$ for $|q| \leq n$. Since the autocorrelation sequence is a complex conjugate symmetric function of n (i.e., $r_x(-n) = r_x^*(n)$), one can readily show that expression (6) can be equivalently represented in the form specified by expression (5) whereby the prevailing coefficients are related by

$$w(n) r_x(n) = \sum_{k=n}^{q} b_k b_{k-n}^* \quad \text{for} \quad 0 \leq n \leq q \qquad (7)$$

Given the q+1 values of the product $w(n)r_x(n)$, one may readily solve this nonlinear system of $q+1$ equations to obtain the MA spectral models b_k coefficients as used in expression (5). Expressions (6) and (7) then constitute a systematic procedure for generating a MA model of time series based on given autocorrelation values.

In most applications, one has available only a set of time series observations (3) (and not autocorrelation values) upon which to generate a MA spectral model. If expression (6) is to be used for this objective, it will then be necessary to obtain estimates of the autocorrelation elements from the given time series observations. The unbiased estimator as specified by

$$\hat{r}_x(n) = \frac{1}{N-n} \sum_{k=1}^{N-n} x(n+k) x^*(k) \qquad 0 \leq n \leq q \qquad (8)$$

is often used for this purpose in which it is assumed that the MA model order is such that $q<N$. Alternatively, the periodogram method has served as a useful procedure for effecting a MA spectral estimate from a set of contiguous time series observations [6]. The periodogram possesses the additional advantage of being efficiently implemented by the fast Fourier transform algorithm. The inherent order of a MA periodogram model is N-1.

III. AR SPECTRAL MODELING

In this section, the method of linear prediction is used for generating an AR spectral model associated with a given time series. In particular, the coefficients of the p^{th} order AR spectral model as specified by

$$\hat{S}_x(e^{j\omega}) = \left| \frac{b_0}{1 + a_1 e^{-j\omega} + \ldots + a_p e^{-jp\omega}} \right|^2 \qquad (9)$$

will be determined by solving a system of p+1 linear equations in the p+1 coefficient unknowns $a_1, a_2, \ldots, a_p, b_0$. These equations are obtained by considering the specific problem of predicting the time series element x(n) by a linear combination of the p most recent time series elements $x(n-1), x(n-2), \ldots, x(n-p)$. It will turn out that the resultant set of optimal equations thus obtained will correspond exactly with those equations which arise when using the

"maximum entropy" method of spectral estimation. This equivalency has been previously recognized [7].

As indicated, the task at hand is that of estimating the time series element x(n) by means of the linear combination

$$\hat{x}(n) = -\sum_{k=1}^{p} a_k x(n-k) \tag{10}$$

in which the optimal prediction coefficients a_k° are to be ultimately used in the AR spectral model (9). The prediction error is formally given by

$$\begin{aligned} e(n) &= x(n) - \hat{x}(n) \\ &= x(n) + \sum_{k=1}^{p} a_k x(n-k) \end{aligned} \tag{11}$$

It is now desired to select the complex valued a_k prediction coefficients so that the expected value of the error's magnitude squared is minimized. One may straightforwardly show that this mean squared error is given by

$$\begin{aligned} E\{|e(n)|^2\} &= r_x(0) + \sum_{k=1}^{p} [a_k r_x(-k) + a_k^* r_x(k)] \\ &+ \sum_{k=1}^{p} \sum_{m=1}^{p} a_k a_m^* r_x(m-k) \end{aligned} \tag{12}$$

Upon using standard calculus methods, it is found that the minimizing predictor coefficients must satisfy the following system of p linear equations

$$\sum_{k=1}^{p} a_k^\circ r_x(m-k) = -r_x(m) \quad \text{for} \quad 1 \le m \le p \tag{13}$$

If this optimal selection is inserted into expression (12), the minimum mean squared error is found to be

$$E\{|e^\circ(n)|^2\} = r_x(0) + \sum_{k=1}^{p} a_k^\circ r_x(-k) \tag{14}$$

In summary, the optimal predictor coefficients are obtained upon solving the system of linear equations (13), and, the corresponding minimum mean squared error is computed by means of relationship (14). From a computational viewpoint, however, a more efficient method for iteratively obtaining the optimal predictor performance is obtained by incorporating relationships (13) and (14) into the single expression

$$\begin{bmatrix} r_x(0) & r_x(-1) & \cdots & r_x(-p) \\ r_x(1) & r_x(0) & \cdots & r_x(-p+1) \\ \vdots & \vdots & & \vdots \\ r_x(p) & r_x(p-1) & \cdots & r_x(0) \end{bmatrix} \begin{bmatrix} 1 \\ a_1^\circ \\ a_2^\circ \\ \vdots \\ a_p^\circ \end{bmatrix} = \begin{bmatrix} E_p \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{15}$$

in which $E_p = E\{|e^\circ(n)|^2\}$. One may solve this Toeplitz system of equations using the Levinson algorithm in a computationally efficient order update manner (e.g., see ref. [7]). Namely, the optimal coefficients of the $p+1^{st}$ order predictor may be recursively obtained from the optimal coefficients of the p^{th} order predictor. As indicated previously, the system of equations (15) is identical to that obtained when using the maximum entropy method of spectral estimation.

If the optimal predictor is performing its objective, it follows that the prediction element $\hat{x}(n)$ will contain all which is predictable in x(n). As such, the prediction error (11) is white like in behavior and its spectral density is then given by $S_e(e^{j\omega}) = E_p$. This behavior is of course dependent on making the order parameter p sufficiently large so as to achieve the desired perfect prediction. Assuming this prediction behavior, it then follows from relationship (11) that the spectral density function of the time series {x(n)} is given by expression (9) in which the coefficient $b_0^2 = E_p$ and the a_k coefficients are obtained upon solving relationship (13) or equivalently relationship (15).

In this analysis, it has been assumed that one has available the time series autocorrelation values $r_x(0), r_x(1), \ldots, r_x(p)$. More realistically, such perfect autocorrelation knowledge is almost never available. In a typical application, one has available only a sampled set of time series observations as exemplified by expression (3). If the AR spectral estimation procedure as represented by expression (15) is to be incorporated, one could use the given time series observations to obtain estimates of the required autocorrelation elements. Alternatively, a set of deterministic prediction error equations can be minimized so as to obtain the prediction coefficients. In effect, this is the approach usually taken in evolving the Burg algorithm [7].

IV. ARMA SPECTRAL MODELING

The time series {x(n)} is said to be an ARMA process of order (p,q) if it is generated according to the linear causal relationship

$$x(n) + \sum_{k=1}^{p} a_k x(n-k) = \sum_{k=0}^{q} b_k w(n-k) \tag{16}$$

where the excitation sequence {w(n)} is a zero mean white noise time series whose individual elements have variance one. It is a simple matter to show that the spectral density function associated with this response time series {x(n)} is given precisely by expression (4). Thus, there is an equivalency between a rational spectral density model and the response of a linear system to a white noise excitation.

a_k Coefficient Determination

A procedure for identifying an ARMA model's a_k autoregressive coefficients involves examining its second order statistical characterization. This is achieved by first multiplying both sides of

expression (16) by $x^*(n-m)$ and then taking expected values. The results of this operation are the well known Yule-Walker equations

$$r_x(m) + \sum_{k=1}^{p} a_k r_x(m-k) = 0 \qquad \text{for } m > q \tag{17}$$

where it is important to note that the lag variable m is here restricted to be larger than the ARMA model's numerator order q.[1]

In effect, the above Yule-Walker equations indicate that an ARMA time series' autocorrelation elements are interrelated in a well defined linear manner for appropriate lag values. This observation then provides the vehicle for determining the ARMA model's associated a_k coefficients. To be more specific, let us now express the first t Yule-Walker equations (17) in the following matrix format

$$\begin{bmatrix} r_x(q) & r_x(q-1) & \cdots & r_x(q+1-p) \\ r_x(q+1) & r_x(q) & \cdots & r_x(q+2-p) \\ \vdots & & & \\ r_x(q+t-1) & r_x(q+t-2) & \cdots & r_x(q+t-p) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_p \end{bmatrix} = - \begin{bmatrix} r_x(q+1) \\ r_x(q+2) \\ \\ r_x(q+t) \end{bmatrix} \tag{18}$$

or in the more compact representation

$$R\,\underline{a} = -\underline{r} \tag{19}$$

where R is a t×p matrix and $\underline{a}$ and $\underline{r}$ are p×1 and t×1 vectors, respectively.

It is readily shown that this system of equations will have a unique solution provided that $t \geq p$. To obtain the ARMA model's a_k coefficients, one then simply solves this system of linear equations. From a computational viewpoint, it is convenient to set t equal to its minimum value of p. For reasons which will be spelled out in Section VI, however, it will often be advantageous to let t take on values exceeding p.

b_k Coefficient Determination

To determine the ARMA model's b_k coefficients, it will be expedient to introduce the autocorrelation sequence's causal image

$$r_x^+(n) = r_x(n)u(n) - \tfrac{1}{2} r_x(0)\delta(n) \tag{20}$$

where $u(n)$ and $\delta(n)$ denote the standard unit-step and Kronecker delta sequences, respectively. The autocorrelation sequence may be recovered from its causal image through the relationship

$$r_x(n) = r_x^+(n) + r_x^+(-n)^* \tag{21}$$

[1] The Yule-Walker equations associated with lag values $0 \leq m \leq q$ will involve the ARMA model's b_k coefficients in a nonlinear manner.

whose validity is established by making use of the complex conjugate symmetry property of autocorrelation sequences (i.e., $r(-n) = r^*(n)$). The Fourier transform of expression (21) is found to be

$$S_x(e^{j\omega}) = S_x^+(e^{j\omega}) + S_x^+(e^{j\omega})^* \tag{22}$$

in which $S_x^+(e^{j\omega})$ denotes the Fourier transform of the autocorrelations' causal image (20). In what is to follow, a systematic procedure for identifying $S_x^+(e^{j\omega})$ will be given which, with the utilization of expression (22), results in the overall times series' spectral density.

This identification is perhaps best achieved by introducing the following auxiliary sequence

$$c(n) = r_x^+(n) + \sum_{k=1}^{p} a_k r_x^+(n-k) \tag{23}$$

in which the a_k coefficients as generated from expression (19) are used. Due to the nature of the causal image sequence and the underlying Yule-Walker equations (17), it is readily shown that this auxiliary sequence is identically zero outside the indexing range $0 \leq n \leq \max(p,q)$. Using this fact, the Fourier transform of relationship (23) is found to yield

$$C_s(e^{j\omega}) = \sum_{n=0}^{s} c(n)e^{-j\omega n} \qquad , s = \max(p,q) \tag{24}$$

$$= A_p(e^{j\omega})S_x^+(e^{j\omega}) \tag{25}$$

Using this result in equation (22), the required ARMA spectral density formulation is obtained

$$S_x(e^{j\omega}) = \frac{A_p(e^{j\omega})C_s^*(e^{j\omega}) + A_p^*(e^{j\omega})C_s(e^{j\omega})}{A_p(e^{j\omega})A_p^*(e^{j\omega})} \tag{26}$$

To obtain the ARMA model's b_k coefficients, we next incorporate relationships (4) and (26) to generate the complex conjugate symmetrical polynomial (in the $e^{j\omega}$) expression

$$B_q(e^{j\omega})B_q^*(e^{j\omega}) = A_p(e^{j\omega})C_s^*(e^{j\omega}) + A_p^*(e^{j\omega})C_s(e^{j\omega}) \tag{27}$$

A spectral factorazation of the right side polynomial will yield 2q roots which occur in complex conjugate reciprocal sets. One then need only select an appropriate q of these roots to determine the required $B_q(e^{j\omega})$ term (e.g., the minimum phase selection).

To summarize, the spectral density function and the model coefficients corresponding to an ARMA time series of order (p,q) may be obtained by following the systematic procedure outlined in Table 1. To carry out this process, it is necessary to have knowledge of the order pair (p,q) and the autocorrelation elements $r_x(n)$ for $0 \leq n \leq q+p$.

1. Solve Relationship (19) for the p autoregressive a_k coefficients. This will require setting $t \geq p$.
2. Generate the auxiliary sequence c(n) and its Fourier transform using expressions (23) and (24), respectively.
3. The required spectral density is given by relationship (26).
4. Perform a spectral factorization of the polynomial $B(e^{j\omega})B^*(e^{j\omega})$ as given by expression (27) to obtain the required b_k coefficients.

Table 1: Generation of the spectral density and the ARMA model parameters associated with a given set of autocorrelation values.

V. EXAMPLE

To demonstrate the relative effectiveness of the MA, AR, and ARMA modeling schemes presented in the last three sections, let us consider the classical problem of generating spectral estimates of a time series composed of two sinusoids in additive white noise. Namely, the time series will be governed by the relationship

$$x(n) = a_1 \sin(\omega_1 n + \theta_1) + a_2 \sin(\omega_2 n + \theta_2) + w(n) \qquad (28)$$

in which θ_1 and θ_2 are independent random variables uniformly distributed on $[0, 2\pi]$ and w(n) is a zero mean white noise process whose individual elements have variance σ^2. The sinusoidal amplitudes a_1 and a_2, and, normalized frequencies ω_1 and ω_2 will be here taken to be unknown constants. [2]

It is readily shown that the autocorrelation sequence corresponding to this time series is given by

$$r_x(n) = \frac{a_1^2}{2}\cos(\omega_1 n) + \frac{a_2^2}{2}\cos(\omega_2 n) + \sigma^2\delta(n) \qquad (29)$$

Upon taking the Fourier transform of this autocorrelation sequence, the associated spectral density function is found to be

$$S_x(e^{j\omega}) = \frac{a_1^2}{2}\pi[\delta(\omega-\omega_1)+\delta(\omega+\omega_1)] + \frac{a_2^2}{2}\pi[\delta(\omega-\omega_2)+\delta(\omega+\omega_2)] + \sigma^2 \quad \text{for } -\pi \leq \omega \leq \pi \qquad (30)$$

This spectral density function is seen to be composed of dirac delta functions located at frequencies $\pm\omega_1$ & $\pm\omega_2$ riding on top of a constant value σ^2 due to the additive white noise.

Using the given autocorrelation elements (29) as entries, the three spectral estimation procedures just described were next utilized to generate MA, AR, and ARMA models. The specific choices for the time series parameters were taken to be $a_1 = \sqrt{20}$, $\omega_1 = 0.4\pi$, $a_2 = \sqrt{2}$, $\omega_2 = 0.426\pi$ and $\sigma^2 = 1$. This selection provides individual sinusoid signal-to-noise ratios of 10dB (decibels) and 0dB. Due to the relative closeness of the sinusoid frequencies, this example provides an excellent measure of the frequency resolution capabilities of the three modeling procedures. A brief description of the results obtained for this example now follows.

MA Spectral Estimates

The autocorrelation elements as specified by expression (29) where incorporated into the MA spectral model relationship (6) in which the window function is taken to be rectangular (i.e., $w(n) = 1$ for $0 \leq n \leq q$). The spectral estimates thereby achieved for the specific order selections q = 15, 30, and 200 are displayed in Figure 1. From these plots it is clear that the MA modeling procedure is unable to resolve the sinusoids for the order selections q = 15 and 30. When q is set to 200, it is possible to just barely detect the presence of the lower amplitude sinusoid. It is apparent from this example that classical Fourier approaches provide relatively poor vehicles for achieving frequency resolution even when exact autocorrelation elements are used.

AR Spectral Estimates

The autocorrelation elements (29) were next incorporated into the optimum one-step predictor (or maximum entropy) expression (15), with AR order choices of p = 15 and 30. The two AR spectral estimates which resulted are shown in Figure 2 where it is apparent that a frequency resolution is achieved for p = 30, but, not for p = 15. In contrast to the classical Fourier approach, the AR spectral modeling procedure is capable of achieving the required frequency resolution with a reasonably small order model.

ARMA Spectral Estimate

In the final modeling approach, the autocorrelation elements (29) were next used to obtain an ARMA model of order (4,4) using the procedure as outlined in Table 1. The resultant spectral estimate is plotted in Figure 3 and is seen to correspond precisely with that as given in equation (30).[3] This should not be surprising considering the fact that the given autocorrelation sequence (29) is an ARMA time series of order (4,4). Thus, the model used in this case precisely matches the time series being examined.

VI. HIGH PERFORMANCE METHOD OF ARMA SPECTRAL MODELING

It is possible to adapt many of the ideas of Section IV to achieve an ARMA spectral estimate when only the time series observation (3) are

[2] The normalized frequency variables are taken to lie in the interval $[0, \pi]$.

[3] If infinite precision computations were utilized, there would be an exact pole-zero cancellation and an associated constant spectral density function.

available (and not autocorrelation values). We shall again treat separately the cases of autocorrelation and moving average coefficient determination.

Autoregressive Coefficient Estimation

To implement the autoregressive coefficient selection process as represented by relationship (19), it will be necessary to compute appropriate autocorrelation estimates from the given set of time series' observations. The high performance ARMA method effects these estimates in the guise of a convenient matrix format which lends itself to a particularly efficient computational realization [2]&[4]. In particular, the autocorrelation matrix and vector required in expression (19) are estimated according to

$$\hat{R} = Y^{\dagger}X \tag{31}$$

$$\hat{\underline{r}} = Y^{\dagger}\underline{x} \tag{32}$$

where the dagger symbol † denotes the operation of complex conjugate transposition. The $(N-p)\times p$ Toeplitz type matrix X is specified by

$$X = \begin{bmatrix} x(p) & x(p-1) & \cdots & x(1) \\ x(p+1) & x(p) & \cdots & x(2) \\ \vdots & \vdots & & \vdots \\ x(N-1) & x(N-2) & & x(N-p) \end{bmatrix} \tag{33}$$

while the $(N-p)\times t$ Toeplitz type matrix Y has the form

$$Y = \begin{bmatrix} x(p-q) & x(p-q-1) & \cdots & x(p-q-t+1) \\ x(p-q+1) & x(p-q) & \cdots & x(p-q-t+2) \\ \vdots & \vdots & & \vdots \\ x(N-q-1) & x(N-q-2) & \cdots & x(N-q-t) \end{bmatrix} \tag{34}$$

and $\underline{x}$ is a $(N-p)\times 1$ vector given by[4]

$$\underline{x} = [x(p+1),\ x(p+2),\ \ldots\ x(N)]' \tag{35}$$

In formulating matrix Y, we have used the convention of setting to zero any elements x(k) for which k lies outside the observation index range $1 \le k \le N$.

If the autocorrelation matrix and vector estimates (31) and (32), respectively, are substituted into the Yule-Walker relationship (19), however, it is generally found that the resultant system of t equations in the p autoregressive coefficients is inconsistent for $t > p$. This is due to inevitable inaccuracies in the autocorrelation estimates, and, to a possible improper ARMA model order choice. In any case, the system of equations with these estimate substitutions will give rise to the $t\times 1$ Yule-Walker approximation error vector as specified by

$$\underline{e} = Y^{\dagger}X\,\underline{a} + Y^{\dagger}\underline{x} \tag{36}$$

Upon taking the expected value of $\underline{e}$, it is found that for the ARMA modeling order choice in which $q \ge p$, that this expectation results in

$$E\{e(k)\} = (N-q-k)\left[r_x(q+k) + \sum_{m=1}^{p} a_m r_x(q+k-m)\right],\ 1 \le k \le t \tag{37}$$

while for the modeling order case $q < p$ this expectation produces

$$E\{e(k)\} = \begin{cases} (N-p)\left[r_x(q+k) + \sum_{m=1}^{p} a_m r_x(q+k-m)\right], & 1 \le k \le p-q \\ (N-q-k)\left[r_x(q+k) + \sum_{m=1}^{p} a_m r_x(q+k-m)\right], & p-q < k \le t \end{cases} \tag{38}$$

In either ordering case, it is seen that when the time series is an ARMA process of order (p,q), the expected value of the error vector $\underline{e}$ can be made equal to zero by a proper choice of the autoregressive coefficient vector $\underline{a}$. Namely, this selection would be such that the underlying Yule-Walker equations (17) are satisfied.[5] This implies that the system of equations (36) with $\underline{e} = \underline{\theta}$ provides an unbiased and a consistent estimate of the Yule-Walker equations.

With the above thoughts in mind, an appealing approach to selecting the autoregressive coefficient vector is immediately suggested. Namely, $\underline{a}$ is chosen so as to make the error vector "as close" to its expected value of $\underline{\theta}$ as possible. This is of course predicated on the assumption that the time series is an ARMA process of order (p,q) or less. In order to attain a tractable procedure for selecting an appropriate autoregressive coefficient vector, we shall introduce the following quadratic functional

$$f(\underline{a}) = \underline{e}^{\dagger}\Lambda\,\underline{e} \tag{39}$$

in which Λ is a $t\times t$ positive-definite diagonal matrix with diagonal elements λ_{kk} that is introduced in order to provide one with the option of weighting differently the various error vector components. It is a simple matter to show that an autoregressive coefficient vector which will render this quadratic functional a minimum must satisfy

$$X^{\dagger}Y\,\Lambda\,Y^{\dagger}X\,\underline{\hat{a}}^{\circ} = -\,X^{\dagger}Y\,\Lambda\,Y^{\dagger}\underline{x} \tag{40}$$

[4] A more generalized version of this estimation scheme can be obtained by substituting the integer k for p wherever p appears in relationships (33) - (35). For ease of presentation, k is here restricted to be p.

[5] A little though will convince oneself that this same conclusion will be reached if both q and p are at least equal to the numerator and denominator orders, respectively, of the underlying ARMA time series.

One then simply solves this consistent system of p linear equations in the p unknown autoregressive coefficients to obtain an estimate for the denominator dynamics of the ARMA model.

Moving - Average Coefficient Estimation

There exist several procedures for estimating the ARMA model's moving average coefficients. We shall now briefly describe two procedures which have provided satisfactory performance and in a sense complement one another.

(i) c_k Method

The procedure which has provided the best frequency resolution behavior is a direct adaption of the c_k method as described in Section IV (and ref. [3]). In particular, using the set of autoregressive coefficient estimates as obtained from expression (40) and a suitable set of autocorrelation estimates $\hat{r}_x(n)$ for $n = 0,1,\ldots,\max(q,p)$ one computes the $\hat{c}(n)$ coefficients using expression (23). These coefficients are then used to achieve the desired ARMA spectral estimate when incorporated into relationship (24) and ultimately relationship (26). Although providing an excellent frequency resolution behavior, this procedure suffers the drawback of not having a guaranteed nonnegative definite spectral density function.[6] It is with this in mind that the following well known smoothed periodogram method was adapted [13].

(ii) Smoothed Periodogram Method

In the smoothed periodogram approach, one first computes the so-called "residual" time-series elements according to the relationship (see ref. [3]).

$$\varepsilon(n) = x(n) + \sum_{k=1}^{p} \hat{a}_k^{\circ} x(n-k) \qquad \text{for } p < n \leq N \qquad (41)$$

in which the $\hat{a}_k^{\circ}$ autoregressive coefficients as obtained by solving expression (40) are incorporated. From this relationship the spectral density expression directly follows

$$S_x(e^{j\omega}) = \frac{S_\varepsilon(e^{j\omega})}{|\hat{A}_p^{\circ}(e^{j\omega})|^2} \qquad (42)$$

If $S_x(e^{j\omega})$ is to correspond to an ARMA spectral model of order (p,q), it is clear that a q^{th} order MA spectral estimate for the residual spectral density $S_\varepsilon(e^{j\omega})$ must be obtained and then substituted into relationship (42). The smoothed periodogram has been found to be a useful tool for this purpose.

In the smoothed periodogram method, one first partitions the computed residual elements (41) into L segments each of length q+1 as specified by

$$\varepsilon_k(n) = \varepsilon(n+p+1+kd) \qquad 0 \leq n \leq q \qquad (43)$$
$$0 \leq k \leq L-1$$

where "d" is a positive integer which specifies the time shift between adjacent segments. These individual segments will overlap if $d \leq q$ and will perfectly partition the residual sequence when $d = q+1$. In order to include only computed elements, the relevant parameters must be selected so that $q + p + 1 + (L-1)d \leq N$. Next the periodogram for each of these L segments is taken and then averaged to obtain the desired q^{th} order smoothed periodogram, that is

$$\hat{S}_\varepsilon(e^{j\omega}) = \frac{1}{L} \sum_{k=0}^{L-1} \left\{ \frac{1}{q+1} \left| \sum_{n=0}^{q} w(n)\varepsilon_k(n)e^{-j\omega n} \right|^2 \right\} \qquad (44)$$

where w(n) is a window sequence that is normally selected to be rectangular (i.e., $w(n) = 1/\sqrt{q+1}$ for $0 \leq n \leq q$). It is readily shown that this procedure results in a desired nonnegative q^{th} order MA spectral density estimate. Unfortunately, its frequency resolution capability is generally not of the same quality as that of the c_k method.[7] On the other hand, the smoothed periodogram method provides more smoothly behaved spectral estimates which contain fewer spurious effects.

To summarize, the required ARMA spectral model is obtained by following the systematic procedure outlined in Table 2. The numerator dynamic estimation procedure to be used will of course depend on the particular characteristic being sought (e.g., frequency resolution, smoothness, etc.)

1.	Specify values for the ARMA model's order parameter pair (q,p), the Yule-Walker equation parameter t, and, the weighting matrice's diagonal elements λ_{kk}.
2.	Using the time series observations x(1), x(2),...,x(N), construct the matrices X, Y, and vector $\underline{x}$ according to relationships (33), (34), and (35), respectively.
3.	Determining the model's autoregressive coefficients by solving relationship (40).
4.	The numerator's dynamics are obtained by using either the (i) c_k method, or, (ii) the smoothed periodogram method.

Table 2: Basic steps of the standard high performance ARMA spectral estimation method: The Block Processing Mode.

The improved spectral estimation performance obtained in using this high performance method over contemporary ARMA techniques such as the Box-Jenkins method is, to a large extent, a consequence of selecting the integer t to be larger than the minimal value p. With the corresponding larger set of Yule-Walker equations that are

[6] This shortcoming may be superficially avoided by taking the absolute value of the spectral estimate.

[7] A similar approach shares the same attributes as does the smoothed periodogram [14].

thereby being approximated, it intuitively follows that the model's autoregressive coefficients will be less sensitive to autocorrelation estimate errors which are embodied in $Y^{\dagger}X$ and $Y^{\dagger}\underline{x}$ than would be the case if t were set to p (as in the Box-Jenkins method). This anticipated improvement in spectral estimation behavior when using the high performance method has in fact been realized on a rather large number of numerical examples [1]-[4]. It is shown in reference [4] & [8] that this high performance method also lends itself to a particular fast adaptive implementation mode when t = p. With the two attributes of improved spectral estimation performance and computational efficiency, this new procedure promises to be an important spectral estimation tool.

It is of interest to note that when q = 0 and t = p, the high performance ARMA spectral estimation method reduces to the well known AR covariance method. Moreover, upon letting t exceed p, the resultant set of expanded AR Yule-Walker equation approximations will typically result in better spectral estimates than the standard AR covariance method. To the author's knowledge, this approach has not been used in the various AR spectral estimation procedures developed to date.

VII. CONCLUSION

A computationally efficient closed form method of ARMA spectral estimation has been presented. It is predicated on the approximation of a set of Yule-Walker equation estimates which are generated from a given set of time series observations. The ARMA model's autoregressive coefficients are determined by solving a consistent system of linear equations.

The spectral estimation performance of this ARMA modeling procedure has been empirically found to exceed that of such counterparts as the maximum entropy and Box-Jenkins methods (e.g., see refs. [1]-[4] & [9]). This behavior is, to a large extent, a consequence of the fact that more than the minimal number of Yule-Walker equation estimates are being approximated to obtain the resultant ARMA model parameters.

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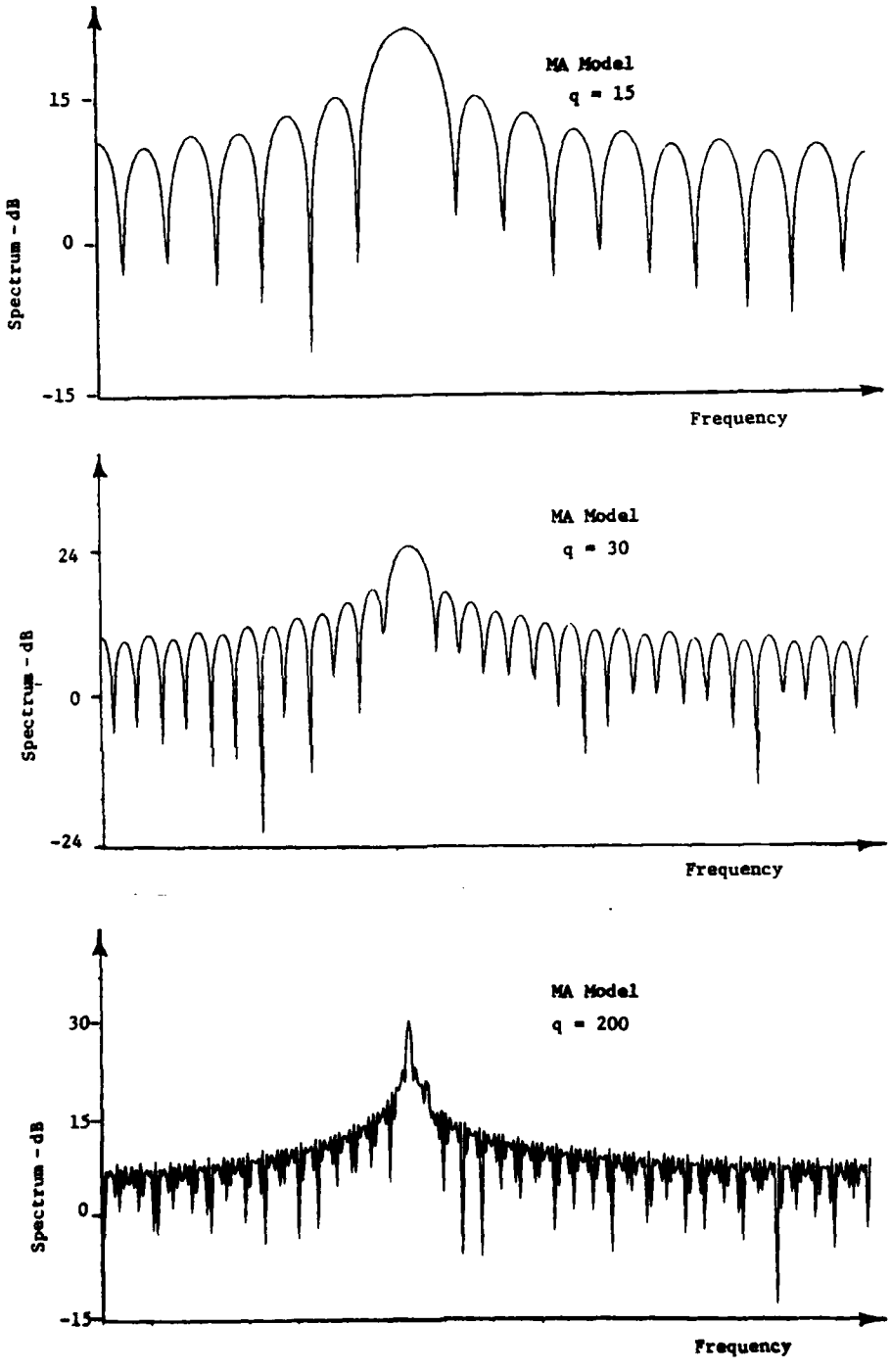

Figure 1: Classical Fourier Method with rectangular window and MA orders (a) q = 15, (b) q = 30, and (c) q = 200.

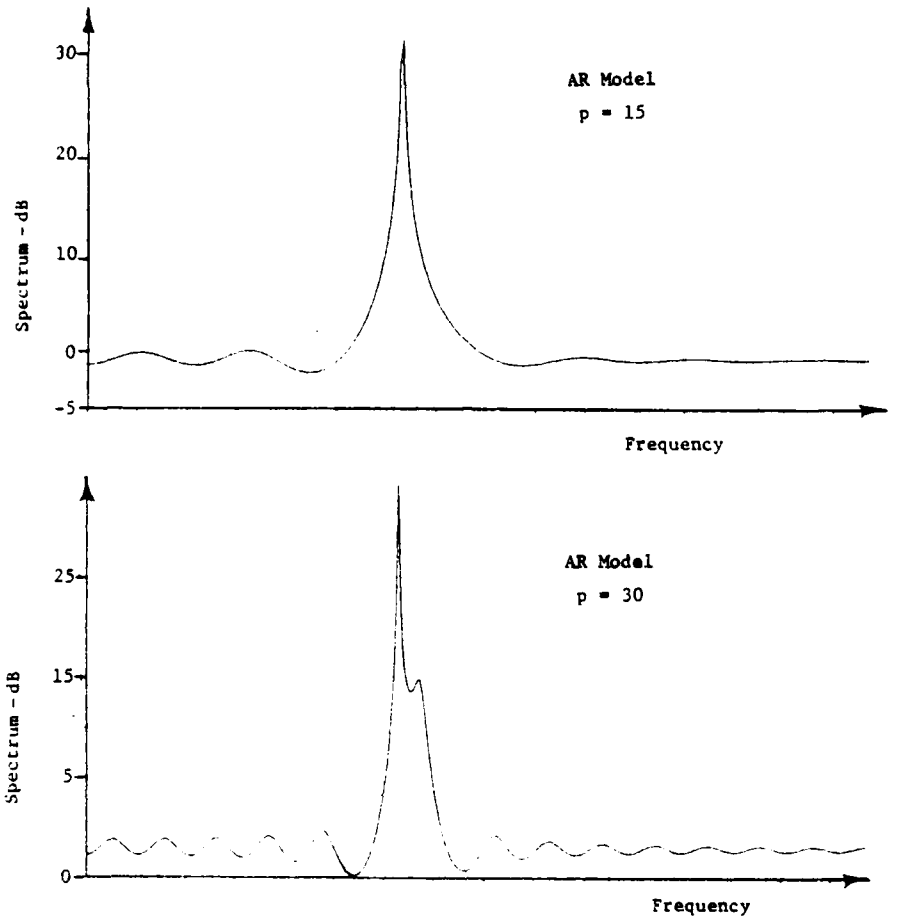

Figure 2: Maximum Entropy Method with AR orders (a) p = 15, (b) p = 30

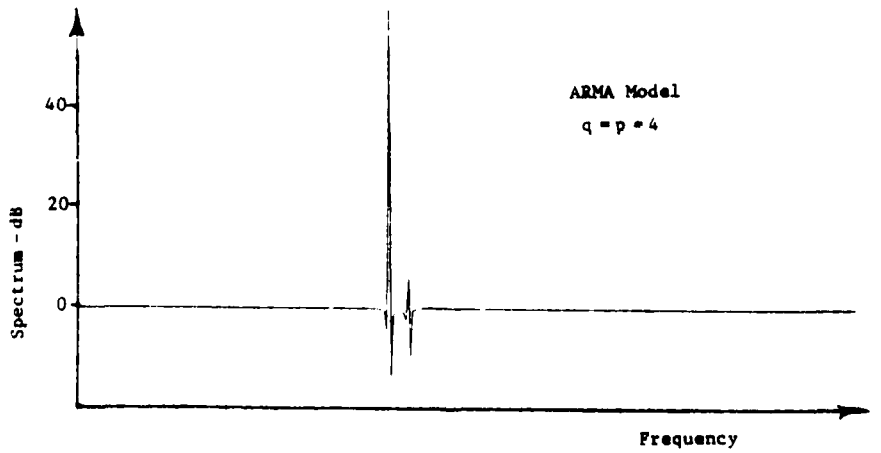

Figure 3: High performance ARMA model of order (4,4)